

# Introduction into the literature of cooperative game theory with special emphasis on dynamic games and the core

**Péter Szikora**

Óbuda University, Keleti Faculty of Business and Management, Institute for Organisation and Management

szikora.peter@kgk.uni-obuda.hu

*Cooperative games model situations where the actors can collaborate, can form coalitions. There exist many static models, however models are too simplistic compared to our more complex world. Despite the fact that there have been several experimental studies on coalition formation there are only very few theoretical papers dealing with the problem in a dynamic context. These papers are not only few in number, but the presented concepts are poorly related. Present paper discusses the process of dynamic coalition formation, and gives a basic insight into the mainstream literature of the quest for finding the core*

*Keywords: game theory, cooperative games, dynamic coalition, equilibrium, core*

*JEL code: C71, C72, C73*

## 1 Introduction

Decisions embedded in a social setting have been discussed and researched for ages. Game theory is the scientific field to deal with decisions of this kind. Models – owing to their nature – are the simplification of the complex situation trying to provide a structural overview or a better understanding of a problem.

The notion of game theory involves static and dynamic, cooperative and non-cooperative models. Present paper endeavours to introduce the basic literature of the cooperative game theories and dedicates special attention to papers on dynamic cooperative theories of game that center on the problem of finding the core. The paper is an introduction to the mainstream concepts, does not aim to evaluate the papers presented.

## 2 What is cooperative game?

The focus of the cooperative game theory is on the problem of revenue distribution (or cost allocation) among many participants (called players) in a joint project. The implicit assumption in a cooperative game is that players can form coalitions and make binding agreements on how to distribute the gains of these coalitions. In a cooperative game the focus is on payoffs and coalitions, rather than on strategies.

The difference between cooperative and non-cooperative game theory is that in a cooperative game binding agreements between players are possible and coalitions are characterized, more abstractly, by the outcomes and payoffs that they can reach. In a non-cooperative game players have explicit strategies. Some examples of cooperative games are e.g.: *three cooperating cities*, *the glove game*, *permutation game*, *voting game*. The cooperative models have an axiomatic normative flavour, in contrast to the equilibrium analysis of non-cooperative theories.

## 3 Dynamic cooperative models of coalition formation

Cooperative games are dynamic, when the players' decisions are modelled in time as a process, where  $N$  is a set of players and  $X$  a set of states. Suppose that for each state in  $X$  and each coalition  $S$  (a nonempty subset of  $N$ ), a possible set of "coalitional moves" (by  $S$ ) to some subset of states is given.

The process of coalition formation is an equilibrium if at any date and at any going state, a coalitional move to some other state can be "justified" by the very same scheme applied in future. The coalition that moves must have higher present value (starting from the state it moves to) for each of its members, compared to (one period) inaction under the going state. In the most general form that we study it, a process of coalition formation precipitates a Markov process on  $X$  (in Konishi and Ray, 2003) the uncertainty reflecting both the choice of the deviating coalition at some state (there may be several potential deviants) and the choice of state that the coalition deviates to (there may be several potential moves). At the same time, we do restrict the class of moves by requiring that for each coalition, moves must be Pareto-efficient for members of that coalition, under the value functions induced by the overall process of coalition formation.

Since the nature of coalition formation is so complex that it is not easy to model it, there has been various approaches to model the process of it. In present chapter three various models are presented.

1, Agastya (1997) introduced a learning model, where players adaptively learn how to bargain. He proved that stochastically stable allocations are a subset of the core.

2, Arnold and Schwalbe's paper (2001) presents a dynamic model of endogenous coalition formation in cooperative games with transferable utility. They wanted to show how coalitions form.

3, Konishi and Ray (2003) studied coalition formation as an ongoing, dynamic process, with payoffs generated as coalitions form, disintegrate, or regroup. They defined the process of coalition formation as an equilibrium if a coalitional move to some other state can be "justified" by the expectation of higher future value, compared to inaction.

Agastya (1997) was one of the first who used a dynamic learning model. He said "Players are myopic and learn to play this game through a process of social learning". He considered a very simple bargaining process, in which each player announces his demand, i.e. the payoff he aspires to get. If there exists a coalition structure such that the vector of all players' demands is feasible, then each player will get his demand with probability one. This way, Agastya entirely abstracts from coalition formation, and focuses on allocations: "it is reasonable to assume that eventually, a maximal coalition (in terms of set inclusion) whose demands are feasible forms" (Agastya, 1999). Indeed, for the class of superadditive games considered by Agastya, this assumption is reasonable.

The second model to be introduced was Tone Arnold and Ulrich Schwalbe's paper (2001). Their paper proposed a dynamic process of endogenous coalition formation in cooperative games. Coalition membership and the allocation of payoffs in each period were determined by a simple adaptation rule that was based on myopic best replies on the part of the players, and players experiment with suboptimal strategies whenever there is a chance that this might lead to a preferred coalition structure. The new aspect introduced in their paper is the explicit formulation of a bargaining process that simultaneously determines the coalition structure and the allocation in each time period. They had an idea to use dynamic learning processes to cooperative games. Hence, they followed Agastya's footsteps, however they departed from Agastya's original model in several aspects. The most important is, that the bargaining process considered by Agastya is simplistic, opposing this: Arnold and Schwalbe's (2001) model is not restricted to superadditive games. They accept the probability of an entity, e.g. a large organizations to operate less efficiently than the sum of its constituent parts. In this case, it is not reasonable to assume that a maximal coalition will form. Instead, they model the coalition formation process explicitly, by the players' choosing both a demand and a coalition in each period. The coalition structure in

each period is thus endogenously determined, which allows them to study how coalitions of players evolve over time. Arnold and Schwalbe's model of boundedly rational play sustains the core as the unique limit state of a Markov chain based on myopic best replies and experimenting on the parts of the players. This results from independent interest and justifies the core on the basis of bounded rationality. In Arnold and Schwalbe's model, a core allocation will be reached even if the Markov process exhibits several inefficient absorbing states (always presuming that the core is non-empty).

Hence, Arnold and Schwalbe's (2001) model is applicable to a wider range of economic problems, such as local public good economies, or clubs, where individuals care not only about allocations but also about the number and/or the characteristics of people in their coalition, since they let players choose their coalitions.

Konishi and Ray (2003) studied coalition formation as an ongoing, dynamic process, with payoffs generated as coalitions form, disintegrate, or regroup. In contrast to Arnold and Schwalbe's (2001) bounded rationality. In Konishi and Ray's model players are farsighted, as, the core will be reached only if it is the unique limit state of the process. This is why, comparing Arnold and Schwalbe's results to that of Konishi and Ray, it shows that bounded rationality may even be precipitous to reaching efficient allocations, as compared to farsightedness.

## 4 The aim of dynamic cooperative models - the core

The core is surely the most widely used cooperative game theory solution concept for games in coalitional form. The core is a set of non-dominated allocations. In other words, the core is exactly the  $(x, P)$  a set of outcomes, which

$$\sum x_i \geq v(S) \quad \forall S \subseteq N.$$

Results might be gained by dynamic approaches where the solution is considered as the ergodic set of the game. This is essentially what happens in Shenoy's (1979) dynamic case, Packer's (1981) stochastic solution, Sengupta, Sengupta's (1994) viable proposals (Proposals viable) and the minimum dominating set's case (Kóczy and Lauwers, 2004). These solutions usually will not be empty. The latter two are especially interesting since they coincide with the nonempty core, however, unfortunately, if the core is empty, the set of viable proposals is too large and difficult to use as a solution.

A core allocation of a coalitional game with transferable utility (TU game) guarantees that each coalition of players obtains at least what it could gain on its own. This means that once players receive a core allocation, no coalition has an incentive to deviate from it.

On the other hand, in case a non-core allocation  $x$  is proposed as a solution for the game, a coalition  $S$  whose members can gain by forming  $S$  may threaten to split off from the grand coalition and thereby suggest an alternative allocation  $y$  according to its myopic preferences. Again, if  $y$  stands outside the core, another blocking coalition could replace  $y$  with a third allocation, and so forth. This process of core finding has been researched by numerous researchers, some of whom further developed each other's model, as in case of Kóczy (2006) and Sengupta and Sengupta (1996).

Sengupta and Sengupta (1996) address the issue for the class of games with non-empty cores. They adopt the notion of weak dominance to formulate the possible transitions among allocations and show that the core, if non-empty, is accessible in the sense that it can be reached from any non-core allocation through a sequence of successive blocks

Kóczy (2006) addresses the issue of estimating the number of blocks needed to reach the core. Based on the accessibility of the core proved by Sengupta and Sengupta (1996), he shows that the core can be reached via a *bounded* sequence of blocks and proposes an algorithm for generating such a sequence.

In the following an example is provided to create a better understanding of this process. Considering a three-player TU-game where the grand coalition obtains 3, pairs get 2, singletons get 0 the core consists of a single payoff-vector  $(1, 1, 1)$ .

Let  $a^k = ((1/k), 2 - (1/k), 1)$  and  $b^k = ((1/k), 1, 2 - (1/k))$ , consider an arbitrary  $m \in \mathbb{N}^+$ , and the process:  $a^m \rightarrow b^{m-1} \rightarrow a^{m-2} \rightarrow b^{m-3} \rightarrow \dots \rightarrow b^1 = a^1 = (1, 1, 1)$ .

This process terminates in the core in exactly  $m - 1$  steps. As  $m$  is arbitrarily large, the number of steps to reach the core via such a path has no upper bound. (Figure 1)

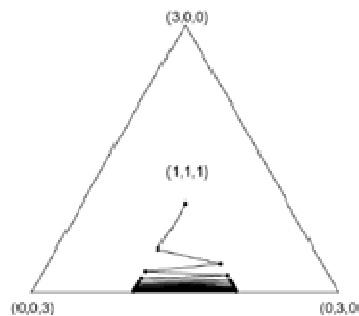


Figure 1 A finite, but unbounded sequence of dominance (From Koczy,2006)

Kóczy and Lauwers (2004) prove that the coalition structure core consisting of all absorbing outcomes, if non-empty, is a minimal accessible set. On the other hand, they note that an analogous result holds for the collection of accessible outcomes in a dual sense, namely, the set of accessible outcomes, if non-empty, is a minimal absorbing set. The question is when would a game possess an accessible outcome. In the search for an existence theorem for accessible outcomes, Yang (2011) focuses on the class of cohesive games and shows that every cohesive game with no absorbing outcome must possess accessible outcomes.

Kóczy and Lauwers (2007) also prove that each game generates a unique minimal dominant set and argue that the minimal dominant set is a non-empty extension of the coalition structure core. (An accessible set is said to be a dominant set if it is also an absorbing set.) They prove that in case the game under consideration is cohesive, (i) the minimal dominant set coincides with either the coalition structure core or the set of absorbing outcomes; (ii) the minimal dominant set can be reached from any outcome in linear time.

To determine the length of the core finding process, Yang (2010) and Béal et al. (2010), respectively, give alternative upper bounds for the number of blocks needed to access the core. The former is described as the number of active coalitions, 3 and the latter is equal to the number of pairs of players, i.e.,  $n(n-1)/2$ , where  $n$  is the cardinality of the player set. Nevertheless, Béal et al. (2013), found another upper bound  $(n^2+4n)/4$ , which is lower than this.

Kóczy and Lauwers (2007) also introduce a certain similarity relation on the outcome space and show that the accessibility of the coalition structure core can be met in a bounded number of blocks, where the bound is given in terms of the number of similarity classes and is at least exponential in the number of players.

To sharpen the result, Yang (2011) shows that the bound can be reduced significantly. The coalition structure core can be reached from any outcome in quadratic time.

The emptiness of the core has kept the researchers interested from the beginning. Bondareva (1963) and Shapley (1967) had set up the conditions of the non-empty core independently. During that time another research had begun about a similar but non-empty solution. Zhou (1994) listed the three points of the requirements. One solution is never empty, its not defined to the players either on a predefined nor on all the possible partition.

In Yang's paper (2010), they first attempt to extend Sengupta and Sengupta's analysis to incorporate games with empty cores with the aid of an alternative notion of dominance relation,  $z$ -dominance, in which a payoff vector  $z = (z_i)$  is adopted as a reference point which guarantees that each player  $i$  outside the blocking coalition obtains at least  $z_i$ .

They study conditions for an allocation to be an accessible allocation, i.e., an allocation that can be reached from any allocation via a sequence of successive transitions, and prove that a game possesses an accessible allocation if and only if the core is a singleton or an empty set. Finally, they apply the analysis to address the accessibility of the foregoing core-extensions by showing that when the core is empty, each allocation in each of these core-extensions is an accessible allocation.

## Conclusions

The field of game theory is ever developing, with novelties for those interested in each and every year. However these fields still require deeper research and better, more suitable models to create better understanding. Present paper attempted to generate interest in game theory and provide a basic introduction into the mainstream papers on collaborative dynamic processes of finding the core.

## References

- [1] Agastya, M., 1997. Adaptive play in multiplayer bargaining situations. *Review of Economic Studies* 64, 411–426.
- [2] Agastya, M., 1999. Perturbed adaptive dynamics in coalition form games. *Journal of Economic Theory* 89, 207–233.
- [3] Beal, S., Remila, E., Solal, P., 2010. On the number of blocks required to access the core. MPR Paper 26578, University Library of Munich, Germany.
- [4] Beal, S., Remila, E., Solal, P., 2013. Accessibility and stability of the coalition structure core. *Mathematical Methods of Operations Research*, 1-16
- [5] Bondareva O.N. 1963. Some Applications of Linear Programming Methods to the Theory of Cooperative Games. *Problemy Kybernetiki* 10. 119-139.
- [6] Einy, E., Holzman, R., Monderer, D., 1999. On the least core and the Mas-Colell bargaining set. *Games Econ. Behav.* 28, 181–188.
- [7] Kóczy, L, Lauwers, L, 2004. The coalition structure core is accessible. *Games and Economic Behavior Elsevier*, vol. 48, 86-9.
- [8] Kóczy, L, 2006. The core can be accessed in a bounded number of blocks. *Journal of Mathematical Economics* 43, 56-64.

- [9] Kóczy, L., Lauwers, L., 2007. The minimal dominant set is a nonempty core extension. *Games and Economic Behavior* 61, 277–298.
- [10] Konishi, H., Ray, D., 2003. Coalition Formation as a Dynamic Process, *Journal of Economic Theory* 110, 1-41.
- [11] Mas-Colell, A., 1989. An equivalence theorem for a bargaining set. *J. Math. Econ.* 18, 129–139.
- [12] Packel, E.W., 1981. A stochastic solution concept for n-person games. *Mathematics of Operations Research* 6, 349–363.
- [13] Roth, A.E., vande Vate. J. H., 1990 Random paths to stability in two-sided matching, *Econometrica* 58. 1475-1480
- [14] Sengupta A, Sengupta K., 1996. A property of the core. *Games and Economic Behavior* 12 Nr 2., 266-273.
- [15] Shenoy P.P., 1979. On Coalition Formation: A Game-Theoretical Approach. *International Journal of Game Theory* 8, 133-164.
- [16] Yang, Y, 2010. On the accessibility of the core, *Games and Economic Behavior* 69, 194-199.
- [17] Yang, Y, 2011. Accessible outcomes versus absorbing outcomes, *Mathematical Social Sciences* 62, 65-70.
- [18] Yang, Y, 2012. On the accessibility of the core-extensions, *Games and Economic Behavior* 74, 687-698.
- [19] Zhou L., 1994. A New Bargaining Set of an N-Person Game and Endogenous Coalition Formation. *Games and Economic Behavior*.6, 512-526.