

# Dynamic cooperative models of coalition formation and the core

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*Cooperative games model situations where the actors can collaborate, can form coalitions. We know many static models, but our world is more complex. Despite the fact that there have been several experimental studies on coalition formation there are only very few theoretical papers dealing with the problem in a dynamic context. These papers are not only few in number, but the presented concepts are poorly related. In this paper I discuss two approaches: I explain the process of dynamic coalition formation, and I look at and review a learning model. After it I show some paper about the core. The main aspect is how can reach the core.*

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## 1 Introduction

What is cooperative game? The implicit assumption in a cooperative game is that players can form coalitions and make binding agreements on how to distribute the proceeds of these coalitions. What is the process of coalition formation? Let  $N$  be a set of players and  $X$  a set of states. Suppose that for each state in  $X$  and each coalition  $S$  (a nonempty subset of  $N$ ), a possible set of “coalitional moves” (by  $S$ ) to some subset of states is given.

In a cooperative game the focus is on payoffs and coalitions, rather than on strategies. The prevailing analysis has axiomatic normative flavour, in contrast to the equilibrium analysis of non-cooperative theory.

The difference between cooperative and non-cooperative game theory is that a cooperative game binding agreements between players are possible in a non-cooperative game players have explicit strategies, whereas in a cooperative game players and coalitions are characterized, more abstractly, by the outcomes and

payoffs that they can reach. Some examples of cooperative games are e.g.: *three cooperating cities, the glove game, permutation game, voting game.*

Let's see a cooperative game example. An example for the voting game: The United Nations Security Council consists of fifteen members (five permanent and 10 others). Motions must be approved by nine members, including all the permanent members. This situation gives rise to a 15-player so called voting game. Such games are also called simple games. Coalitions with worth equal to 1 are called winning, the other coalitions are called losing.

A process of coalition formation is an equilibrium if at any date and at any going state, a coalitional move to some other state can be “justified” by the very same scheme applied in future: the coalition that moves must have higher present value (starting from the state it moves to) for each of its members, compared to (one period) inaction under the going state. In the most general form that we study it, a process of coalition formation precipitates a Markov process on  $X$ ; the uncertainty reflecting both the choice of the deviating coalition at some state (there may be several potential deviants) and the choice of state that the coalition deviates to (there may be several potential moves). At the same time, we do restrict the class of moves by requiring that for each coalition, moves must be Pareto-efficient for members of that coalition, under the value functions induced by the overall process of coalition formation.

In the first section I look at and review a selection of such approaches Agastya (1997) introduces a learning model, where players adaptively learn how to bargain. They show that stochastically stable allocations are a subset of the core. Konishi and Ray (2003) studied coalition formation as an ongoing, dynamic process, with payoffs generated as coalitions form, disintegrate, or regroup. They said the process of coalition formation is an equilibrium if a coalitional move to some other state can be “justified” by the expectation of higher future value, compared to inaction. Arnold and Schwalbe's paper (2001) presents a dynamic model of endogenous coalition formation in cooperative games with transferable utility. They want to know how coalitions form.

In the second section I present an comparison of the following papers: Sengupta and Sengupta (1996), Kóczy (2006), Kóczy and Lauwers (2004), Yang (2010) All of these papers have the core in the central. In Sengupta and Sengupta's paper, they showed that for any transferable utility game in coalitional form with nonempty core, given any allocation outside the core, there is an allocation in the core that directly dominates it. Kóczy (2006) addresses the issue of estimating the number of blocks needed to reach the core.

## 2 Dynamic cooperative models of coalition formation

First of all, I examined Tone Arnold and Ulrich Schwalbe's paper(2001). Their paper proposed a dynamic process of endogenous coalition formation in cooperative games. Coalition membership and the allocation of payoffs in each period are determined by a simple adaptation rule that is based on myopic best replies on the part of the players, and players experiment with suboptimal strategies whenever there is a chance that this might lead to a preferred coalition structure. The new aspect introduced in their paper is the explicit formulation of a bargaining process that simultaneously determines the coalition structure and the allocation in each time period. They had an idea to use dynamic learning processes to cooperative games. Agastya was one of the first who use dynamic learning model. He said "Players are myopic and learn to play this game through a process of social learning" and Arnold and Schwalbe followed this model.

Schwalbe's model departs from Agastya in several respects. First, Agastya entirely abstracts from coalition formation, and focuses on allocations. The bargaining process considered by Agastya is simple: Each player announces his demand, i.e. the payoff he aspires to get. If there exists a coalition structure such that the vector of all players' demands is feasible, then each player will get his demand with probability one. Agastya (1999) writes: "it is reasonable to assume that eventually, a maximal coalition (in terms of set inclusion) whose demands are feasible forms." Indeed, for the class of superadditive games considered by Agastya, this assumption is reasonable. Arnold and Schwalbe's model, however, is not restricted to superadditive games. They allow for the case that, e.g. large organizations may operate less efficiently than the sum of their constituent parts. In this case, it is not reasonable to assume that a maximal coalition will form. Instead, they model the coalition formation process explicitly, by the players' choosing both a demand and a coalition in each period. The coalition structure in each period is thus endogenously determined, which allows them to study how coalitions of players evolve over time. Further, letting the players choose their coalitions makes Arnold and Schwalbe's model applicable to a wider range of economic problems, such as local public good economies, or clubs, where individuals care not only about allocations but also about the number and/or the characteristics of people in their coalition.

Konishi and Ray (2003) study coalition formation as an ongoing, dynamic process, with payoffs generated as coalitions form, disintegrate, or regroup. In contrast, Arnold and Schwalbe's model of boundedly rational play sustains the core as the unique limit state of a Markov chain based on myopic best replies and experimenting on the parts of the players. This result is of independent interest, as it justifies the core on the basis of bounded rationality. What is more, comparing Arnold and Schwalbe's result to that of Konishi and Ray, it shows that bounded rationality may even be precipitous to reaching efficient allocations, as compared to farsightedness: When players are farsighted, as in Konishi and Ray's model, the

core will be reached only if it is the unique limit state of the process. With boundedly rational players, as in Arnold and Schwalbe's model, a core allocation will be reached even if the Markov process exhibits several inefficient absorbing states (always presuming that the core is non empty). The main difference to Arnold and Schwalbe's approach is that, in Konishi and Ray's model, the players are farsighted rather than myopic.

### 3 The Core

The core is surely the most widely used solution concept for games in coalitional form. The core is set of non-dominated allocations. In other words, the core is exactly the  $(x, P)$  a set of outcomes, which

$$\sum x_i \geq v(S) \quad \forall S \subseteq N.$$

Same age as the idea of game theory, discovered by Edgeworth [1881] named as "contract curve" (Kannai [1992]). The emptiness of the core has kept the researchers interested from the beginning. Bondareva [1963] and Shapley [1967] had set up the conditions of the non-empty core independently. During that time an other research had begun about a similar but non-empty solution. Zhou [1994] listed the three points of the requirements. One solution is never empty, its not defined to the players either on a predefined nor on all the possible partition.

Results might be given by some dynamic approaches where the solution is considered as the ergodic set of the game. This is essentially what happens in Shenoy [1979] dynamic case, Pachel [1981] stochastic solution, Sengupta, Sengupta [1994] viable proposals (Proposals viable) and the minimum dominating set's case (Kóczy Lauwers, [2002]). These solutions usually will not be empty. The latter two especially interesting to coincide with the nonempty core. Unfortunately, however, if the core is empty, the set of viable proposals is too large and difficult to use as a solution.

Once an allocation in the core is proposed, it is plainly a compelling candidate for a solution: no coalition can improve upon it. Yet this is not a persuasive argument for predicting that an observed allocation ought to be in the core whenever the core is nonempty. Granted that an allocation outside the core is not stable per se since some coalition can improve upon it; but, were the initial proposal an allocation outside the core, why it may be reasonable to suppose that an element in the core would eventually replace it remains unexplained. Sengupta' paper offers one explanation.

Provided that a (transferable utility) game in coalitional form has a core at all, they show that, given any allocation outside the core, an allocation in the core indirectly dominates it. Put differently, an allocation in the core can be reached in

finitely many steps of recontracting from any arbitrary initial proposal via successive domination through counterproposals. Indeed, at each intermediate step some coalition can improve upon the prevailing proposal by an individually rational and efficient counterproposal.

Call a set of allocations  $K$  *indirectly stable* if (i) no allocation in  $K$  indirectly dominates another and (ii) given any allocation not in  $K$ , there is some allocation in  $K$  that indirectly dominates it. Their result shows that the core is indirectly stable. Feldman (1974) and Green (1974) have made important contributions on the subject. In Green's paper, a counterproposal is randomly selected at each step from those that dominate the prevailing proposal; the process is shown to converge to the core with probability one for a large class of games. Feldman's approach is similar: he provides a simpler proof but the analysis is conducted in a setting where the set of feasible allocations is finite.

Kóczy (2006) addresses the issue of estimating the number of blocks needed to reach the core. Based on the accessibility of the core proved by Sengupta and Sengupta (1996), Kóczy shows that the core can be reached via a *bounded* sequence of blocks and proposes an algorithm for generating such a sequence.

Let's see an example to understand this. Consider the three-player TU-game where the grand coalition obtains 3, pairs get 2, singletons get 0. The core consists of a single payoff-vector  $(1, 1, 1)$ .

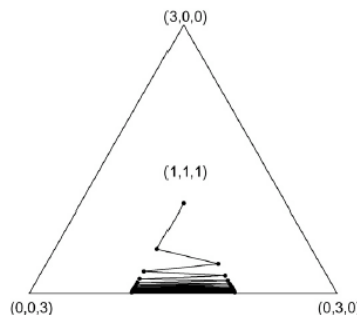


Figure 1

A finite, but unbounded sequence of dominance (From Koczy,2006)

Let  $a^k = ((1/k), 2 - (1/k), 1)$  and  $b^k = ((1/k), 1, 2 - (1/k))$ , consider an arbitrary  $m \in \mathbb{N}^+$ , and the process:  $a^m \rightarrow b^{m-1} \rightarrow a^{m-2} \rightarrow b^{m-3} \rightarrow \dots \rightarrow b^1 = a^1 = (1, 1, 1)$ . This process terminates in the core in exactly  $m - 1$  steps. As  $m$  is arbitrarily large, the number of steps to reach the core via such a path has no upper bound. (Figure 1)

In Yang's paper (2010), they study the accessibility of the core by adopting an alternative notion of dominance, namely, *sequential z-dominance*, which suggests that when contemplating counterproposals, blocking coalitions should employ a predecided core imputation  $z$  as a bench mark and take all former negotiations into consideration such that the blocking imputations chosen at each step of the

bargaining process do not make any of the past blocking coalitions worse off. They prove that when the notion of sequential  $z$ -dominance is employed, *every* sequence of successive blocks surely leads to the core in a *bounded* number of steps. Their bound is much lower than the one provided by Kóczy.

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