



# Risk based capital allocation

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## 1. Abstract

*Several methods are known for the purpose of capital allocation. There are some properties that allocation methods should satisfy. These are full allocation, core compatibility, riskless allocation and suitability for performance measurement (compatible with Return on Risk Adjusted Capital calculation). When we think about practical application we should also consider simplicity of the methods.*

*We examine six methods from the point of view if they are satisfying the mentioned criteria and try to find out if there is any of them fulfilling both five properties. Our main question is the fulfillment of core compatibility that we test using simulation.*

*We find that the Expected Shortfall based Euler method satisfies all the above given criteria but symmetry (that is not anyways desirable as we will point out) and also very easy to calculate. All this enables us to suggest this method for practical applications.*

## 2. Introduction

Capital allocation means the process of distributing the capital to different business lines or portfolio elements. As financial markets are getting more and more rigorously regulated the importance of capital allocation is also increasing. In the article we are going to analyze six possible allocation methods and try to find out if there is any of them that both results allocation satisfying some reasonable requirements and both easy to employ – so it can be advised for practical purposes. As both Hungarian and international studies are showing the capital allocation is not yet a very common practice at financial institutions, finding such a method might be a real benefit. We will show that if we use Expected Shortfall as risk measure Euler method is satisfying the above mentioned criteria: it fulfills the requirements of full allocation, core compatibility, riskless allocation and it is suitable for performance measurement. Furthermore it is really easy to calculate independently from the number of considered business lines.

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### 3. The allocation of capital

To discuss the allocation methods first we have to define what capital is. In our approach capital will be considered as *economic capital*. Let us suppose that  $X_1, X_2, \dots, X_n$  are random variables, standing for the profits on the assets in a portfolio  $N = \{1, 2, 3 \dots n\}$ . Let  $X$  denote the profit on the whole portfolio. Economic capital means the reserve that a financial company needs to cover its losses and it is determined by a risk measure that we are going to denote by  $p$ :

$$EC = p(x)$$

According to Artzner et al. (1997) a risk measure  $p(x)$  is coherent if it satisfies the following properties:

- *Subadditivity*:  $\rho(X + Y) \leq \rho(X) + \rho(Y)$ .
- *Monotonicity*: if  $X \geq^{a.e.} Y$  then  $\rho(X) \leq \rho(Y)$ .
- *Homogeneity of first degree*: for all  $\lambda \geq 0, \lambda \in \mathbb{R}$  we have  $\rho(\lambda X) = \lambda \rho(X)$ .
- *Translation invariance*: for all  $\lambda \in \mathbb{R}$   $\rho(X + \alpha) = \rho(X) - \alpha$ .

The most common risk measure is Value at Risk (VaR). Given the significance level  $\alpha$  it is calculated as:

$$VaR_\alpha(X) = -\inf\{x | P(X \leq x) > \alpha\}.$$

Even though VaR is very popular it's not a coherent measure of risk as it violates the criterion of subadditivity. Because of this we are going to use another risk measure, Expected Shortfall (from now on ES):

$$ES_\alpha = -\frac{1}{\alpha} \int_0^\alpha F^\leftarrow(p) dp,$$

where  $F^\leftarrow(p) = \inf\{x | F(x) \geq p\}$  stands for the generalized inverse function. Given the significance level  $1-\alpha$  ES results the average loss in  $\alpha\%$  of the worst cases (while VaR shows the best outcome among them). Now we can say that an allocation rule is a function that assigns the vector  $p(X|X)$  to the portfolio  $p(X_i|X)$  where the element of the vector denotes the capital allocated to division  $i$ , given a specific risk measure,  $p$ . Similarly to the coherent properties of a risk measure there are some requirements that an allocation rule should satisfy (following Denault, 2001 and Tasche, 2008).

- *Full allocation*:  $\sum_{i \in N} \rho(X_i|X) = \rho(X)$
- *Core compatibility*:  $\sum_{i \in M} \rho(X_i|X) \leq \rho(\sum_{i \in M} X_i)$
- *Symmetry*: If by joining any subset  $M \subset N\{i, j\}$ , portfolios  $i$  and  $j$  both make the same contribution to the risk capital, then  $\rho(X_i|X) = \rho(X_j|X)$
- *Riskless allocation*:  $\rho(\alpha * D) = -\alpha$ , where  $D$  denotes the risk free discount rate

RORAC-compatibility: let us define the RORAC (Return On Risk Adjusted Capital) of the  $i$ th division by:

$$RORAC(X_i|X) = \frac{E(X_i)}{\rho(X_i|X)},$$

where  $E(X_i)$  denotes the expected return on division  $i$ . We can state that the  $p(X_i|X)$  risk contributions are RORAC-compatible if there exists some  $\varepsilon_i > 0$  so that

$$RORAC(X_i|X) > RORAC(X) \Rightarrow RORAC(X + hX_i) > RORAC(X)$$

for all  $0 < h < \varepsilon_i$ . A capital allocation rule is RORAC-compatible if all the risk contributions,  $p(X_i|X), i = 1, 2, \dots, n$  fulfill the above given criterion.

The interpretation of the axioms is as follows. Full allocation represents the natural requirement that the sum of the allocated capital of the subsets must be equal to the risk of the main unit. Core compatibility means that a portfolio should not have any subset that is allocated more capital than the risk capital it would face as a separate entity. Symmetry ensures that the capital allocated to the elements should only depend on its contribution to the risk of the portfolio and nothing else. The meaning of the riskless allocation axiom is also clear: by adding one unit risk free asset (e.g. cash) to the portfolio its risk should be decreased by the same amount. RORAC-compatibility becomes relevant if we would like to use the result of the allocation in performance measurement. It says that if we add a new element to the portfolio that's risk is higher than the risk of the original portfolio then the risk of the new portfolio also has to increase. An allocation rule satisfying core compatibility, symmetry and riskless allocation principles is often referred as coherent (Denault, 2001).

## 4. Capital allocation methods

Let us overview some of the most common capital allocation methods. We are going to present these methods without the specification of the risk measure so they can be easily applied with other kinds of risk measures, not only ES that we are going to use for risk measurement. When we introduce the first five methods we will follow the article of Homburg and Scherpereel [2008] plus we will also analyze the Euler method.

### 4.1 Activity based method

Activity based method is a long-known and popular allocation scheme (see e.g. Hamlen et al., 1977) that allocates the joint capital to the portfolio elements in proportion to their own risk:

$$\rho(X_i|X) = \frac{\rho(X_i)}{\sum_{j=1}^n \rho(X_j)} \cdot \rho(X) \quad (1)$$

A really serious drawback of the method is that it does not consider the dependence structure between the portfolio elements. This means it does not ‘reward’ those elements with smaller units of risk allocated that are correlated negatively with the rest of the portfolio.

## 4.2 Beta method

Beta method – also known as covariance-based allocation – is discussed for example by Panjer [2002]. Let  $Cov(i, N)$  denote the covariance of the portfolio  $N$  and the  $i$ th division. As we know the beta of the  $i$ th division can be calculated as  $\beta_i = \frac{Cov(i, N)}{\sigma(N)^2}$ . This way the allocated risk of  $X_i$  is:

$$\rho(X_i|X) = \beta_i \cdot \rho(X) \quad (2)$$

If we consider a financial institution and we would like to allocate its risk to its divisions we can assume that the weight of each division is one. In this case the sum of the betas equals one, but otherwise we have to modify the method in order to satisfy the full allocation axiom as follows:

$$\rho(X_i|X) = \beta_i \cdot \frac{\rho(X)}{\sum_{i=1}^n \beta_i} \quad (3)$$

## 4.3 Incremental method

For further reading on the method we suggest Jorion [2001]. Let us define the increment caused by a division given the coalition  $K$  as  $\Delta(X_i|K) = \rho(K) - \rho(K \setminus X_i)$  for all coalitions  $K \in N$  and for all  $X_i \in K$  where  $\rho(\emptyset) := 0$ . Now we can define the allocated capital similarly to the activity-level method:

$$\begin{aligned} \rho(X_i|X) &= \frac{\Delta\rho(X_i|N)}{\sum_{j=1}^n \Delta\rho(X_j|N)} \cdot \rho(X) = \\ &= \Delta\rho(X_i|N) + \frac{\Delta\rho(X_i|N)}{\sum_{j=1}^n \Delta\rho(X_j|N)} \cdot \left( \rho(X) - \sum_{j=1}^n \Delta\rho(X_j|N) \right), \end{aligned} \quad (4)$$

where the second definition is only necessary to enable us for easier comparison with the cost gap method.

## 4.4 Cost gap method

After a smaller amendment on the above given method we get the cost gap allocation rule, first introduced by Tijs and Driessen [1986]:

$$\rho(X_i|X) = \begin{cases} \Delta\rho(X_i|N), & \text{if } \rho(X) - \sum_{i=1}^n \Delta\rho(X_i) = 0 \\ \Delta\rho(X_i|N) + \frac{\gamma_i}{\sum_{k=1}^n \gamma_k} \cdot (\rho(X) - \sum_{i=1}^n \Delta\rho(X_i)), & \text{otherwise} \end{cases}$$

where  $\gamma_j = \min_{\emptyset \neq K \subset N, i \in K} \{\rho(K) - \sum_{j \in N} \Delta \rho(X_j | N)\} \geq 0$ . This means that if the sum of the increments equals the whole risk of the portfolio than we already have the allocation. Otherwise by both methods there's a correction factor that guarantees that the full allocation principle is fulfilled and the difference between the two methods is only the way of correction. However – as we will see – this apparently small modification makes one much better in core-allocation percentage than the other.

## 4.5 Shapley method

The Shapley method is a well-known tool in cost allocation games with a number of favorable properties (for example it always results core allocation in case of convex games). Nevertheless a serious drawback of the method is that it can be calculated easily when we have three players but with more calculation becomes fairly complicated (this holds for the incremental and cost gap methods too). Shapley [1953] investigated if there exists a value that represents the utility that is caused for the player by taking part in the game. The rule calculates the cost by the weighted average of cost increments caused by the given division (we still use the same definition of increments given above):

$$\rho(X_i | X) = \sum_{K \subset N, i \in K} \frac{(|K| - 1)! \cdot (n - |K|)!}{n!} \cdot \Delta \rho(X_i | K) \quad (6)$$

for all  $i=1, \dots, n$ , where  $|K|$  denotes the number of divisions in coalition  $K$

## 4.6 Euler- (or gradient) method

Euler (or gradient) method is a very old allocation scheme known from game theory as Aumann-Shapley value (see Aumann and Shapley, 1974). Throughout the discussion we are going to use the following notations. First let us calculate the value of a portfolio by the sum product of the value of its elements and their weights:  $X = Y(u) = Y(u_1, \dots, u_n) = \sum_{i=1}^n u_i Y_i$ . Let  $f_{\rho, Y} = \rho(Y(u))$ . Let  $f_{\rho, Y}$  be positively homogenous (that doesn't mean any restriction as we only concern coherent risk measures). The per unit allocated risk is given by:

$$\rho(Y_i | Y) = \frac{d\rho(Y + hY_i)}{dh} \Big|_{h=0} = \frac{df_{\rho, Y}(1, \dots, 1)}{du_i} \quad (7)$$

assuming that  $f_{\rho, Y}$  is continuously differentiable.

For a continuously differentiable and positively homogenous  $f_{\rho, Y}$  we can apply Euler's theorem:

$$f_{\rho, Y} = \sum_{i=1}^n u_i \frac{df_{\rho, Y}(u)}{du_i} \quad (8)$$

so – getting back to our former notation – the following holds:

$$\rho(X) = \sum_{i=1}^n \rho(X_i|X) = \sum_{i=1}^n u_i \cdot \rho(Y_i|Y). \quad (9)$$

The differentiability of the risk measure Expected Shortfall is discussed by Tasche [2000]. According to his results the derivative – if exists – can be calculated as:

$$\rho(X_i|X) = E(X_i|X \leq F^{\leftarrow}(\alpha)). \quad (10)$$

## 5. Examination of the different allocation methods

We examined if the above given capital allocation methods – choosing ES as risk measure – are satisfying the criteria we discussed earlier. Our main question was the fulfillment of the core allocation criterion. We only took into account allocation rules that satisfy full allocation property so that criterion is fulfilled for sure. In this paper we did not consider symmetry because as Buch and Dorfleitner [2008] showed it implies the linearity of the underlying risk measure in case of the gradient principle (that proved to be far the best method in our simulation). Linear risk measures do not account for diversification although we think that is the main point in capital allocation. The tool of the research was simulation. Our aim was to perform a study similar to that Homburg and Scherpereel [2008] did but we made some modifications as we used ES as risk measure while they chose VaR. The second significant difference was that we used historic method counter to the authors who applied delta-normal method. This means that we generated sample return data and calculated ES from this while Homburg and Scherpereel generated only covariance matrices to calculate VaR.

During the simulation we assumed to have a portfolio consisting of three elements and used two different approaches in the simulation. First we generated return time series using random correlation matrices. As the simplest case of the first approach we used normal distribution, but we also ran the simulation with Student t-distribution as it describes the behavior of real returns better. We generated covariance matrices (more precisely, first Cholesky matrices) and we used them to create the random return time series. This way we got multivariate normal and t-distributed time series. At this stage we could calculate the allocated capital applying the methods discussed above. Finally we examined if the allocations resulted from the different rules were core compatible or not: we calculated the allocated risk to the three- and two-element coalitions and the single assets and examined if there was a coalition (or single element) that's stand-alone risk would be lower than the capital allocated to it. We generated 5000 Cholesky matrices (considering both running time and the stability of the result) and to all of them random time series of 500 elements (that means 2 years data if we assume there are 250 trading days a year).

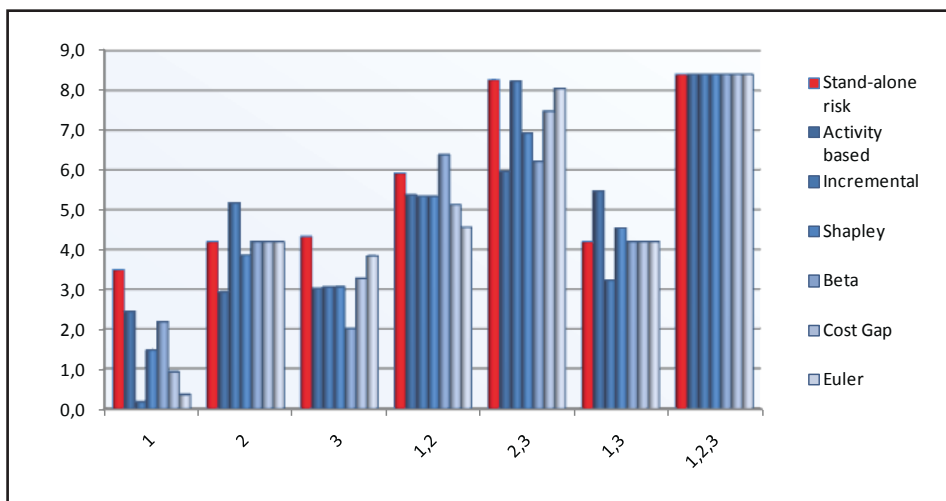
As a second approach we generated return time series from copulas. Because of the presence of simultaneous extreme returns in real life copulas should be even better in modeling financial asset's returns. We chose Clayton-copula for this purpose as it assumes lower tail dependence and that's what we normally observe on financial markets. The

problem we faced was that using Clayton-copula we could only generate positively correlated variables. So we modified this method by multiplying all of the three simulated time series by randomly plus or minus one (we will denote this by Clayton-copula II later). This way we could handle the problem of positive correlation but meanwhile we lost the property that it results higher correlation in periods of turmoil. Even though if for example we look at what has happened when the crisis reached Hungary we could observe that equity and HuF exchange rates were falling dramatically at the same. Considering a portfolio consisting of Hungarian equities and foreign currency the value of the former was steeply decreasing while the other remarkably appreciated. We modeled situations like this with this second method.

Of course, in both cases we wanted to know how many percent of the examined cases resulted core compatible allocation. The closer is the ratio of core compatible outcomes to one the better the method is, naturally.

## 6. Our results

First of all let us present a graph that shows how different methods allocate risk to make the purpose of the simulation more clear (Graph 1). The illustrated case is generated from t-distribution, applying 99% significance level and the following (arbitrarily chosen) correlation coefficients:  $p_{12}=0.5$ ;  $p_{23}=0.5$ ;  $p_{13}=-0.5$ , and standard deviation  $\sigma = 1$  for both of the three assets.



Graph 1: Comparison of the allocations resulted by different methods

In this case Euler- and cost gap methods resulted core compatible allocations but not the others (the graph shows that the risk of the whole portfolio (1,2,3) is the same for all cases as it was computed uniformly with ES). The condition of core compatibility was violated by asset 2 in case of the incremental method, by the coalition consisting of asset 1 and 3

in case of Shapley and activity based method and by coalition consisting of asset 1 and 2 in case of beta method. On the graph this can be observed from the fact that the red column (that shows the stand-alone risk of the asset or coalition) is shorter than the column belonging to a specific method. The graph also shows that in our example Euler and cost gap methods are allocating the risk quite similarly (they both result the same order respect to the value of the allocated capital, but form the remaining four methods only one, the Shapley-method results the same order). This will be in line with our results. So let us now move to them.

We illustrate the results of our simulations together in Table 1. As we mentioned, the closer is the ratio of core compatible outcomes to one the better the method is. We ran every simulation using three different significance levels (84.13%, 95% and 99%) but as we couldn't observe a remarkable difference we will only give the results for 99% here. The table also contains the results of Homburg and Scherpereel [2008] that enables us for easier comparison. They used confidence level  $\alpha=84.13\%$  but as we have mentioned it doesn't mean a significant difference as ratios didn't significantly change when we chose another significance level.

$\alpha=99\%$	Beta method	Cost gap	Shapley	Activity based	Incremental	Euler
Homburg&Scherpereel ( $\alpha=84,13\%$ )	100%	100%	90,1%	66,1%	48,2%	-
Normal distribution	66,2%	99,9%	65,2%	37,8%	22,3%	100,0%
Student-t distribution	55,3%	99,7%	62,9%	36,3%	21,5%	100,0%
Clayton copula I.	83,3%	100,0%	99,6%	95,3%	96,4%	100,0%
Clayton copula II.	76,2%	99,3%	89,3%	70,8%	51,4%	100,0%

Table 1: The ratio of core-compatible allocations

As the table shows we found that cost gap and Euler methods were much better than the other methods, when we employed ES as risk measure. In most cases incremental and activity based methods showed relatively poor performance. The Shapley and the beta methods performed better than the former two that means they resulted core compatible allocations around 60% of the cases. Though thinking about practical use this is still a weak performance.

We also should briefly draw attention to the fact the when we used Clayton copula (I) the results were significantly different from normal or t-distribution: even the worst result was more than 83%. This shows us how dangerous it is to use copulas in modeling. The reason of this is that during risk estimation we concentrate only on the lower tail of the distribution and that is where correlation coefficient is very close to 1 here. This is why we also needed the Clayton copula II denoted running (where we multiplied the time series randomly by 1 and -1).

Getting back to the comparison of the methods our results show that best performing methods were the cost gap and the Euler method. Nevertheless between these two methods there's still a big difference if we also consider how easy the application of a method is – that is also a very important practical standpoint. It is clear that from this point of



view Euler method is the better one, as it can be calculated by the average return of the assets in the  $\alpha\%$  of the worst cases (regarding the whole portfolio). This can be done easily even if we have a large number of portfolio elements. However if we choose cost gap method the increments and correction factors can be easily calculated as long as we consider three or four assets but it becomes very complicated and time-consuming when we want to count it in case of more assets. Furthermore Euler method has other favorable properties too when we use ES or other coherent risk measure. Buch and Dorfleitner [2008] showed that – using gradient method for capital allocation – there are links connecting the axioms of coherent measures of risk and capital allocation rules:

- subadditivity of the risk measure implies core compatibility of the allocation
- translation invariance of the risk measure implies riskless allocation.

As Expected Shortfall is a coherent measure of risk the capital allocation resulted by ES based Euler method also satisfies the axiom of riskless allocation. Nevertheless it is shown by Tasche [2008] that this is the only RORAC-compatible capital allocation method. As we earlier mentioned Buch and Dorfleitner [2008] also showed that symmetry property of the allocation is not by all means desirable as it implies the linearity of the risk measure (if we use the gradient allocation principle) and linear risk measures do not allow for diversification effects. Summarizing all this we can state that method suggested by us (ES based Euler method) has all the properties that are reasonable to expect from an allocation method:

- ES is a coherent measure of risk i.e. it is subadditive, monotone, homogenous of first degree and translation invariant
- the allocation is core compatible, satisfies the riskless allocation axiom and suitable for performance measurement (RORAC-compatible)
- it is easy to apply even if we have an arbitrary large number of divisions.

## 7. Summary

In the paper we tried to find out if there is any capital allocation method that results optimal allocation from every point of view that means it both satisfies the axioms of full allocation, core compatibility, symmetry, riskless allocation and it is also simple enough to be used in real life. The answer is nearly yes as we found a method that satisfies all the above mentioned properties except symmetry. Nevertheless we think that we can be pleased with this result because the symmetry property is not anyways desirable as we mentioned earlier. Our simulation study showed that choosing Expected Shortfall as risk measure and using the Euler method the allocated capital satisfies full allocation property and core compatibility and it is also really simple to use as we only have to calculate the average return of the single asset in those cases when the portfolio's loss exceeds a given level. We also know that this method also satisfies the riskless allocation property and RORAC-compatibility.

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