Mathematical Approach to Everyday Life

Dr Ivana Djolović
idjolovic@tfbor.bg.ac.rs
University of Belgrade, Technical faculty in Bor
Bor, Serbia

Today, in the modern informatical society we are surrounded with different kind of stories in the media related to some predictions, claims, confidence levels and conclusions.

Verbal expressions and everyday phrases are presented to audince in order to warn or just inform people, but mathematics stays in "the backstage". Correctly used mathematics and statistics could be powerful tool for explanation of many situations in everyday life.

This talk will be devoted to some statistical interpretations of real life situation. Starting from some real situation, we will discover where the statistical interpretation is hidden. Also we will emphasize potential traps in understanding the situation.

- ✓ *...9 out of 10 women recommend anti-age cream...*
- ✓ *...30% chance of snow...*
- ✓ *...the average lifetime of a light bulb is 562 days...*
- ✓ *...certain medication is the best solution for headache...*
- ✓ *...6-years old children spend 200 minutes watching TV...*
- ✓ *...less than 5% of our items are defective...*
- ✓ *...washing detergent A is more effective than others...*
- ✓ *...drinking 2 liters of water per day is healthy...*
- ✓ *...100% success in teaching...*

?

- ✓ *Can I beleive in all those numbers?*
- ✓ *How did they get those numbers?*
- ✓ *Real life or suspicious information?*
- ✓ *Who was included in the survey?*

Can we test and check such claims?

Claim: *A 3-month-old babies sleep an avarage 20 hours in a 24 hours.*

Mathematical (statistical) interpretation 1:

A past study claimed that 3-month-old babies sleep an avarage 20 hours in a 24 hours. A researcher took a random sample of 20 babies and obtained that they slept an avarage 19 hours 15 minutes in a 24 hours. Assume that the sleeping times of all 3-month-old babies are normally distributed and population standard deviation is 45 minutes. Using the 5% significance level, test the claim of the earlier study.

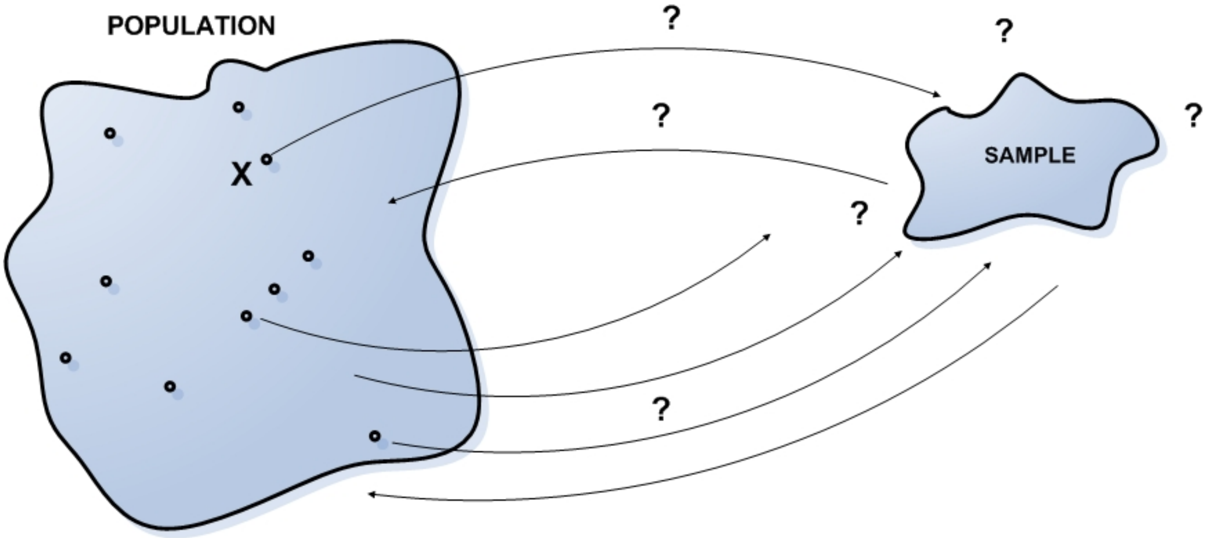
Mathematical (statistical) interpretation 2:

A past study claimed that 3-month-old babies sleep an avarage 20 hours in a 24 hours. A researcher took a random sample of 20 babies and obtained that they slept an avarage 19 hours 15 minutes in a 24 hours with standard deviation of 45 minutes. Assume that the sleeping times of all 3-month-old babies are normally. Using the 5% significance level, test the claim of the earlier study.

The same problem? The same text? NO!!!

Hypothesis Testing - Hypothesis tests about the mean

(hypothesis tests are used to confirm (accept) or deny (reject) a claim that is made about a population)



X – random variable – characteristic
$(x_1,\ x_2,\ x_3,\ \ldots\ x_n)$ *sample*
n – sample size

Population	**Sample**
μ-*population mean* σ-*population standard deviation*	$\bar{x}$-*sample mean* s-*sample standard deviation*

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

$$s^2 = \frac{(x_1-\bar{x})^2 + (x_2-\bar{x})^2 + (x_3-\bar{x})^2 + \ldots + (x_n-\bar{x})^2}{n},\ s = \sqrt{s^2}$$

$$\hat{s}^2 = \frac{(x_1-\bar{x})^2 + (x_2-\bar{x})^2 + (x_3-\bar{x})^2 + \ldots + (x_n-\bar{x})^2}{n-1},\ \hat{s} = \sqrt{\hat{s}^2}$$

Notation:

s - *sample standard deviation*
$\hat{s}$ - *improved sample standard deviation*

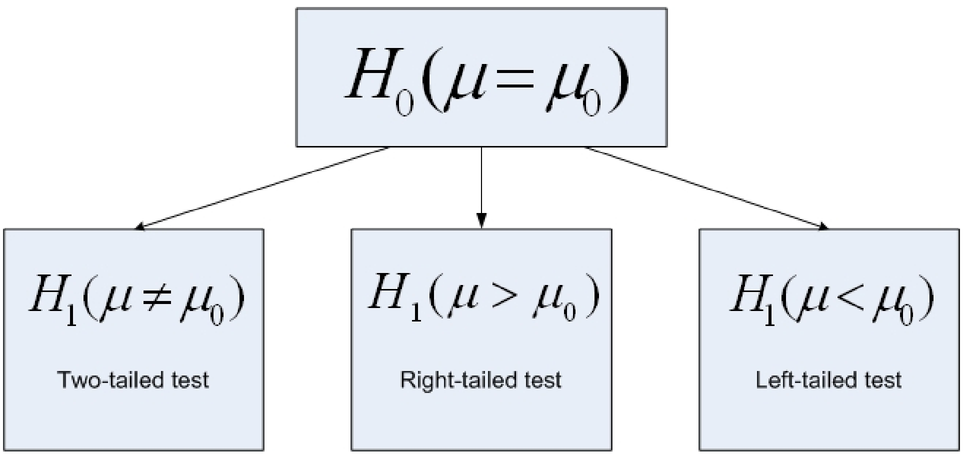
Elements in the hypothesis tests:

- ✓ **Null hypothesis** H_0 *(a claim about a population parameter that is assumed to be true until it is declared false)*
- ✓ **Alternative hypothesis** H_1 *(true if the null hypothesis is false)*

Null hypothesis vs Alternative hypothesis

		Real situation	
		H_0 *true*	H_0 *false*
Decision	*Accept* H_0	*OK*	*Type II Error* β
	Reject H_0	*Type I Error* α	*OK*

- ✓ α ***- the significance level***
- ✓ **C – *the rejection region***
- ✓ ***T – test- statistics*** *(random variable)*
- ✓ ***Statistically significant = Significantly different (****the null hypothesis is rejected; very small probability of happening just by chance;the difference between* $\bar{x}$ *and* μ *is statistically significant***)**
- ✓ ***(Statistically) not significantly different*** *(the difference between* $\bar{x}$ *and* μ *is so small that it may have occured just by chance)*



Hypothesis tests about the mean μ

1. σ known

$X : N\left(\mu, \sigma^2\right)$

Null hypothesis: $H_0\left(\mu = \mu_0\right)$

✓ **Alternative hypothesis:** $H_1\left(\mu \neq \mu_0\right)$

Two-tailed test; the rejection region: $C = \left(-\infty, -z_\alpha\right] \cup \left[z_\alpha, \infty\right)$, *where* $\Phi\left(z_\alpha\right) = \frac{1-\alpha}{2}$;

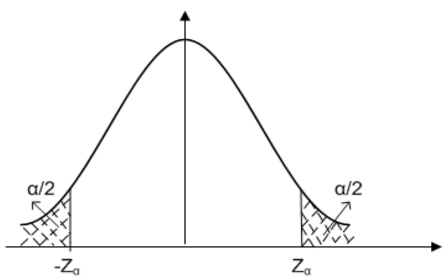
✓ **Alternative hypothesis:** $H_1\left(\mu > \mu_0\right)$

Right-tailed test; the rejection region: $C = \left[z_\alpha, \infty\right)$, *where* $\Phi\left(z_\alpha\right) = \frac{1-2\alpha}{2}$;

✓ **Alternative hypothesis:** $H_1\left(\mu < \mu_0\right)$

Left-tailed test; the rejection region: $C = \left(-\infty, -z_\alpha\right]$, *where* $\Phi\left(z_\alpha\right) = \frac{1-2\alpha}{2}$;

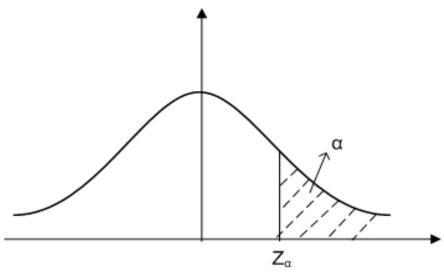
$$T = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \longrightarrow t = \frac{\overline{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$



$\Phi(Z_\alpha) = \frac{1-\alpha}{2}$

$H_o(\mu = \mu_o)$ vs $H_1(\mu \neq \mu_o)$

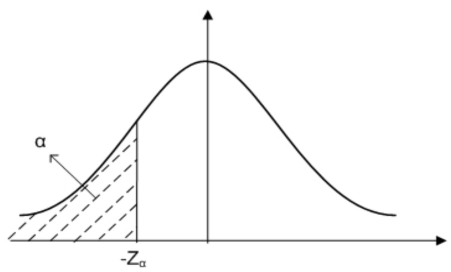
$C = \left(-\infty, -z_\alpha\right] \cup \left[z_\alpha, \infty\right)$



$\Phi(Z_\alpha) = \frac{1-2\alpha}{2}$

$H_o(\mu = \mu_o)$ vs $H_1(\mu > \mu_o)$

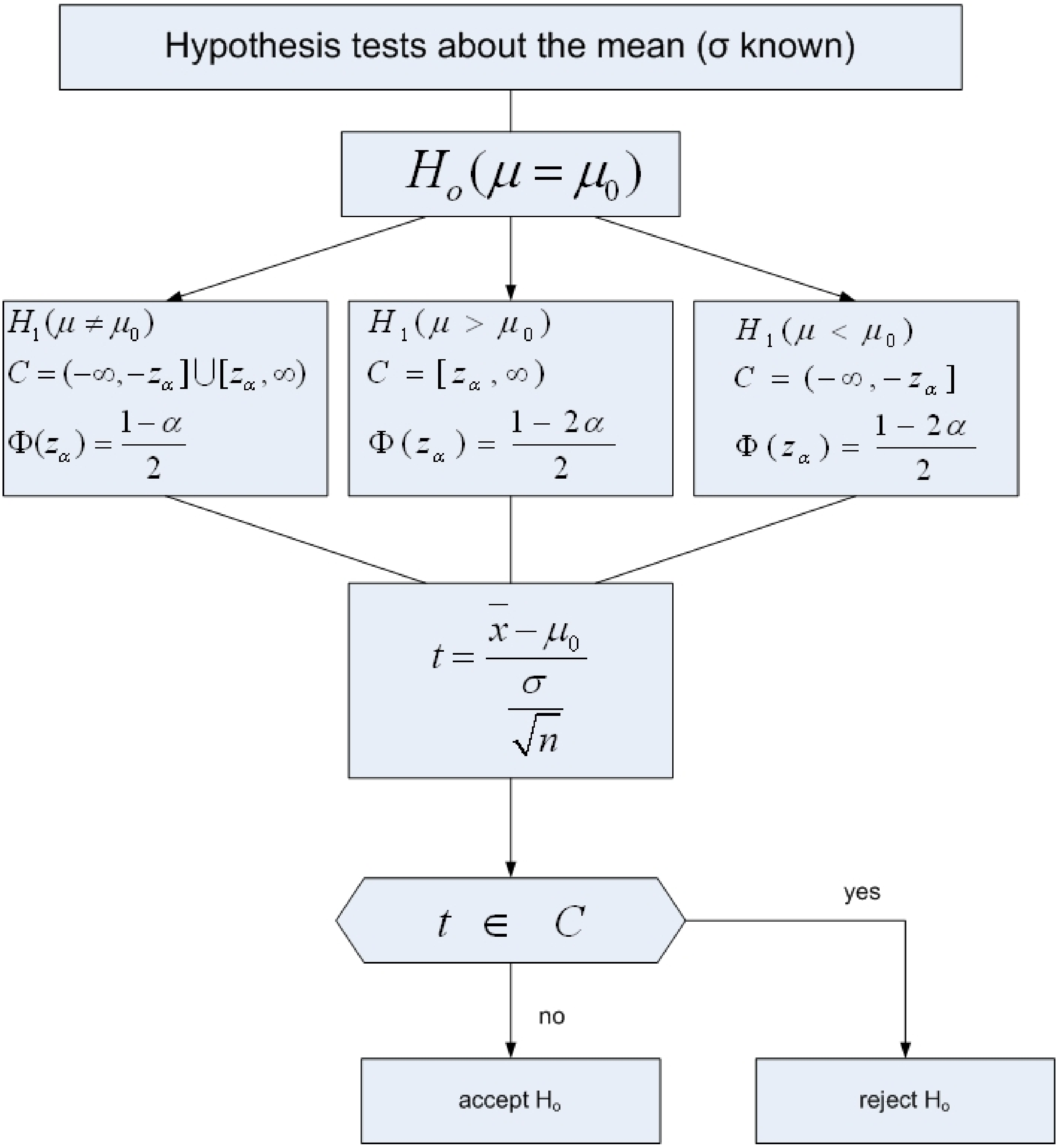
$C = \left[z_\alpha, \infty\right)$



$\Phi(Z_\alpha) = \frac{1-2\alpha}{2}$

$H_o(\mu = \mu_o)$ vs $H_1(\mu < \mu_o)$

$C = \left(-\infty, -z_\alpha\right]$



$\Phi(z_\alpha)$	0.4	0.45	0.475	0.48	0.49	0.495
z_α	1.285	1.645	1.96	2.055	2.325	2.575

Table 1 $$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}} dt$$

x	0	1	2	3	4	5	6	7	8	9
0.0	0.00000	00399	00798	01197	01595	01994	02392	02790	03188	03586
0.1	03983	04380	04776	05172	05567	05962	06356	06749	07142	07535
0.2	07926	08317	08706	09095	09483	09871	10257	10642	11026	11409
0.3	11791	12172	12552	12930	13307	13683	14058	14431	14803	15173
0.4	15542	15910	16276	16640	17003	17364	17724	18082	18439	18793
0.5	19146	19497	19847	20194	20540	20884	21226	21566	21904	22240
0.6	22575	22907	23237	23565	23891	24215	24537	24857	25175	25490
0.7	25804	26115	26424	26730	27035	27337	27637	27935	28230	28524
0.8	28814	29103	29389	29673	29955	30234	30511	30785	31057	31327
0.9	31594	31859	32121	32381	32639	32894	33147	33398	33646	33891
1.0	34134	34375	34614	34850	35083	35314	35534	35769	35993	36214
1.1	36433	36650	36864	37076	37286	37493	37698	37900	38100	38298
1.2	38493	38686	38877	39065	39251	39435	39617	39796	39973	40147
1.3	40320	40490	40658	40824	40988	41149	41309	41466	41621	41774
1.4	41924	42073	42220	42364	42507	42647	42786	42922	43056	43189
1.5	43319	43448	43574	43699	43822	43943	44062	44179	44295	44408
1.6	44520	44630	44738	44845	44950	45053	45154	45254	45352	45449
1.7	45543	45637	45728	45818	45907	45994	46080	46164	46246	46327
1.8	46407	46485	46562	46638	46712	46784	46856	46926	46995	47062
1.9	47128	47193	47257	47320	47381	47441	47500	47558	47615	47670
2.0	47725	47778	47831	47882	47932	47982	48030	48077	48124	48169
2.1	48214	48257	48300	48341	48382	48422	48461	48500	48537	48574
2.2	48610	48645	48679	48713	48745	48778	48809	48840	48870	48899
2.3	48928	48956	48983 !	49010	49036	49061	49086	49111	49134	49158
2.4	49180	49202	49224	49245	49266	49286	49305	49324	49343	49361
2.5	49379	49396	49413	49430	49446	49461	49477	49492	49506	49520
2.6	49534	49547	49560	49573	49585	49598	49609	49621	49632	49643
2.7	49653	49664	49674	49683	49693	49702	49711	49720	49728	49736
2.8	49744	49752	49760	49767	49774	49781	49788	49795	49801	49807
2.9	49813	49819	49825	49831	49836	49841	49846	49851	49856	49861

3.0	0.49865	3.1	49903	3.2	49931	3.3	49952	3.4	49966
3.5	49977	3.6	49984	3.7	49989	3.8	49993	3.9	49995
4.0	499968								
4.5	499997								
5.0	49999997								

$$\Phi(z_\alpha) = 0.475 \Rightarrow z_\alpha = 1.96$$

$$\Phi(z_\alpha) = 0.49! \Rightarrow 2.32 \le z_\alpha \le 2.33 \Rightarrow z_\alpha \approx 2.325 \text{ or } z_\alpha \approx 2.33 \text{ or } z_\alpha \approx 2.32$$

Research 1:

A past study claimed that 3-month-old babies sleep an avarage 20 hours ina a 24 hours. A researcher took a random sample of 20 babies and obtained that they slept an avarage 19 hours 15 minutes in a 24 hours. Assume that the sleeping times of all 3-month-old babies are normally distributed and population standard deviation is 45 minutes. Using the 5% significance level, test the claim of the earlier study.

X - *the sleeping times of all 3-month-old babies;* $X : N\left(\mu,\sigma^2\right)$

$\sigma = 45\min = 0.75h$ (σ *known*) – *population standard deviation*
$n = 20$ *(sample size)*
$\bar{x} = 19h15\min = 19.25h$ *(sample mean)*
$\alpha = 5\% = 0.05$ *(significance level)*

$$\text{Test-statistics } t = \frac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{19.25-20}{\frac{0.75}{\sqrt{20}}} \approx -4.47$$

1) $H_0\left(\mu = 20\right)$ *vs* $H_1\left(\mu \neq 20\right)$
two-tailed test;
the rejection region: $C = \left(-\infty,-z_\alpha\right]\cup\left[z_\alpha,\infty\right)$

$$\Phi\left(z_\alpha\right) = \frac{1-0.05}{2} = 0.475 \Rightarrow z_\alpha = 1.96 \Rightarrow C = \left(-\infty,-1.96\right]\cup\left[1.96,\infty\right)$$

Since $t \approx -4.47 \in C$, *we reject* $H_0\left(\mu = 20\right)$ *- the claim of the earlier study*

2) $H_0\left(\mu = 20\right)$ *vs* $H_1\left(\mu > 20\right)$
right-tailed test;
the rejection region: $C = \left[z_\alpha,\infty\right)$

$$\Phi\left(z_\alpha\right) = \frac{1-2\cdot 0.05}{2} = 0.45 \Rightarrow z_\alpha = 1.645 \Rightarrow C = \left[1.645,\infty\right)$$

Since $t \approx -4.47 \notin C$, *we accept* $H_0\left(\mu = 20\right)$ *- the claim of the earlier study*

3) $H_0\left(\mu = 20\right)$ *vs* $H_1\left(\mu < 20\right)$
left-tailed test;
the rejection region: $C = \left(-\infty,-z_\alpha\right]$

$$\Phi\left(z_\alpha\right) = \frac{1-2\cdot 0.05}{2} = 0.45 \Rightarrow z_\alpha = 1.645 \Rightarrow C = \left(-\infty,-1.645\right]$$

Since $t \approx -4.47 \in C$, *we reject* $H_0\left(\mu = 20\right)$ *- the claim of the earlier study*

1. σ not known

$X : N\left(\mu, \sigma^2\right)$

Null hypothesis: $H_0\left(\mu = \mu_0\right)$

✓ **Alternative hypothesis:** $H_1\left(\mu \neq \mu_0\right)$

Two-tailed test; the rejection region: $C = \left(-\infty, -t_{n-1;\alpha}\right] \cup \left[t_{n-1;\alpha}, \infty\right)$

✓ **Alternative hypothesis:** $H_1\left(\mu > \mu_0\right)$

Right-tailed test; the rejection region: $C = \left[t_{n-1;2\alpha}, \infty\right)$

✓ **Alternative hypothesis:** $H_1\left(\mu < \mu_0\right)$

Left-tailed test; the rejection region: $C = \left(-\infty, -t_{n-1;2\alpha}\right]$

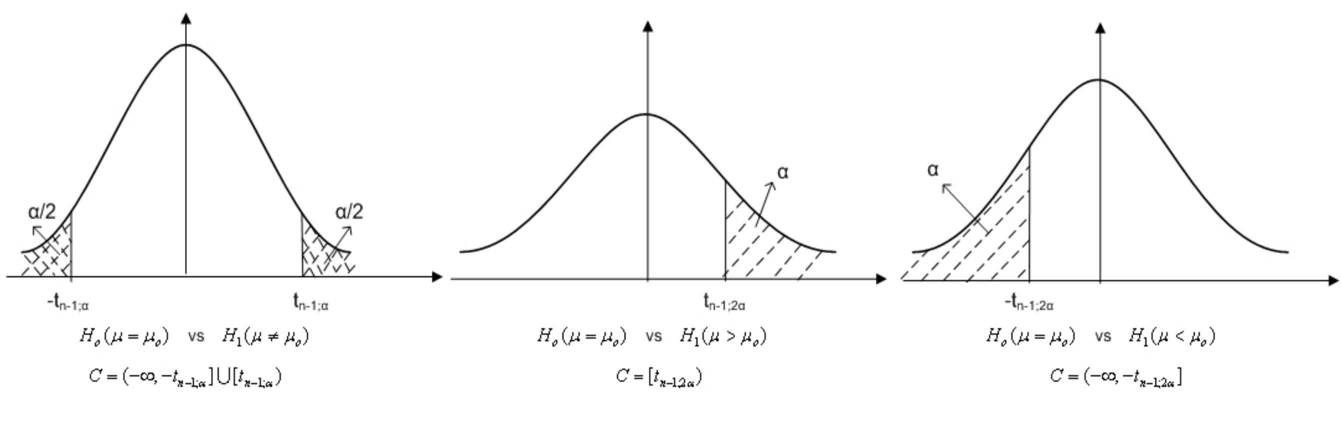
t-distribution (Student's t distribution)

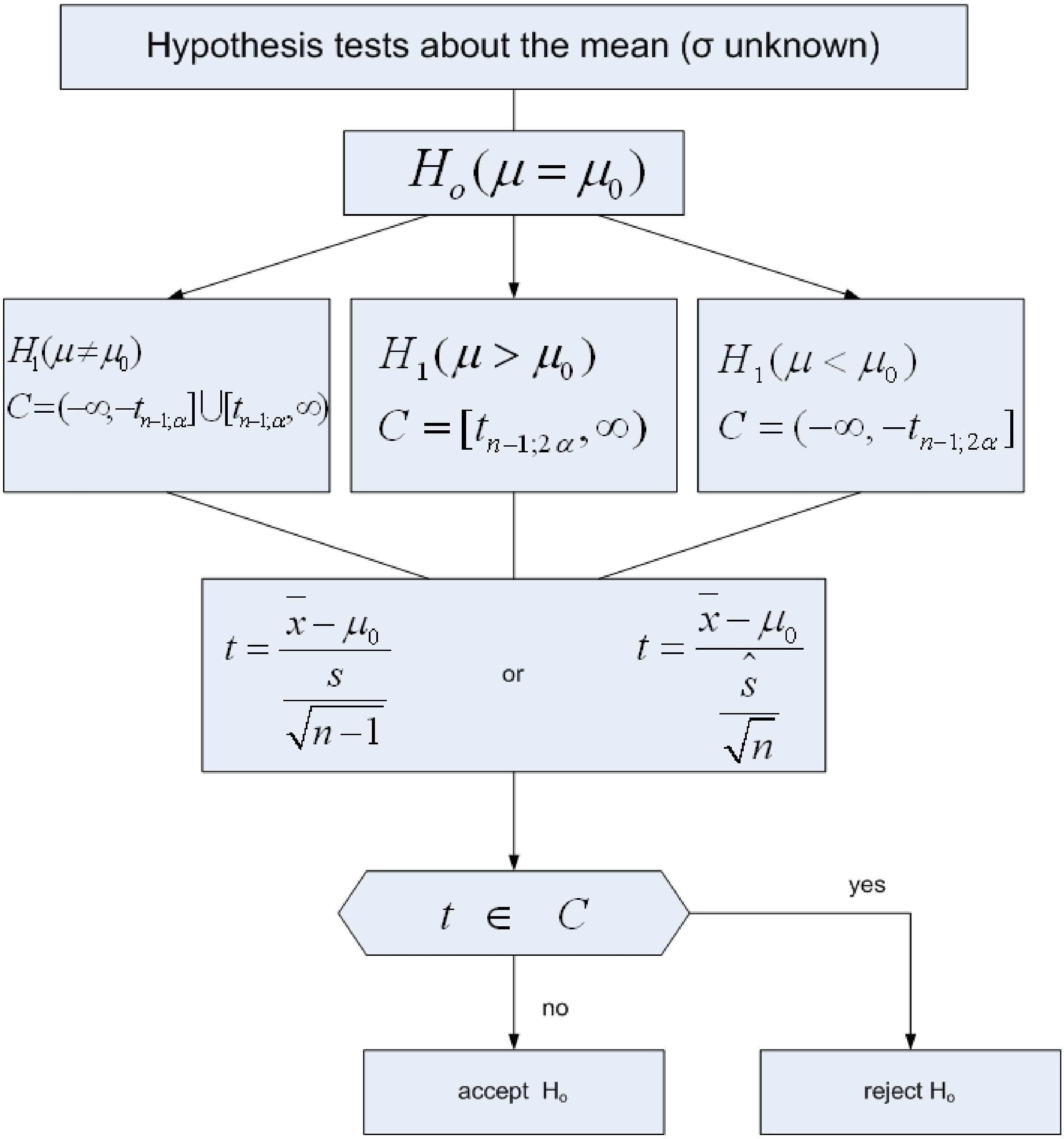
n-1 – degrees of freedom

$$T = \frac{\overline{X} - \mu}{\frac{S}{\sqrt{n-1}}} \longrightarrow t = \frac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n-1}}}$$

OR

$$T = \frac{\overline{X} - \mu}{\frac{\hat{S}}{\sqrt{n}}} \longrightarrow t = \frac{\overline{x} - \mu_0}{\frac{\hat{s}}{\sqrt{n}}}$$





Hypothesis tests about the mean (σ unknown)
$H_o(\mu = \mu_0)$
$H_1(\mu \neq \mu_0)$
$C = (-\infty, -t_{n-1;\alpha}] \cup [t_{n-1;\alpha}, \infty)$
$H_1(\mu > \mu_0)$
$C = [t_{n-1;2\alpha}, \infty)$
$H_1(\mu < \mu_0)$
$C = (-\infty, -t_{n-1;2\alpha}]$
$t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n-1}}}$
or
$t = \frac{\bar{x} - \mu_0}{\frac{\hat{s}}{\sqrt{n}}}$
$t \in C$
yes
no
accept H_o
reject H_o

Table 2 - Student's t-distribution $P\{|t_n| > t_{n;\alpha}\} = \alpha$

	α							
n	0.80	0.60	0.40	0.20	0.10	0.05	0.02	0.01
1	0.325	0.727	1.376	3.078	6.314	12.706	31.821	63.657
2	0.289	0.617	1.061	1.886	2.920	4.303	6.965	9.925
3	0.277	0.584	0.978	1.638	2.353	3.182	4.541	5.841
4	0.271	0.569	0.941	1.533	2.132	2.776	3.747	4.604
5	0.267	0.559	0.920	1.476	2.015	2.571	3.365	4.032
6	0.265	0.553	0.906	1.440	1.943	2.447	3.143	3.707
7	0.263	0.549	0.896	1.415	1.895	2.365	2.998	3.499
8	0.262	0.546	0.889	1.397	1.860	2.306	2.896	3.355
9	0.261	0.543	0.883	1.383	1.833	2.262	2.821	3.250
10	0.260	0.542	0.879	1.372	1.812	2.228	2.764	3.169
11	0.260	0.540	0.876	1.363	1.796	2.201	2.718	3.106
12	0.259	0.539	0.873	1.356	1.782	2.179	2.681	3.055
13	0.259	0.538	0.870	1.350	1.771	2.160	2.650	3.012
14	0.258	0.537	0.868	1.345	1.761	2.145	2.624	2.977
15	0.258	0.536	0.866	1.341	1.753	2.131	2.602	2.947
16	0.258	0.535	0.865	1.337	1.746	2.120	2.583	2.921
17	0.257	0.534	0.863	1.133	1.740	2.110	2.567	2.898
18	0.257	0.534	0.862	1.330	1.734	2.101	2.552	2.878
19	0.257	0.533	0.861	1.328	1.729	2.093	2.539	2.861
20	0.257	0.533	0.860	1.325	1.725	2.086	2.528	2.845
21	0.257	0.532	0.859	1.323	1.721	2.080	2.518	2.831
22	0.256	0.532	0.858	1.321	1.717	2.074	2.508	2.819
23	0.256	0.532	0.858	1.319	1.714	2.069	2.500	2.807
24	0.256	0.531	0.857	1.318	1.711	2.064	2.492	2.797
25	0.256	0.531	0.856	1.316	1.708	2.060	2.485	2.787
26	0.256	0.531	0.856	1.315	1.706	2.056	2.479	2.779
27	0.256	0.531	0.855	1.314	1.703	2.052	2.473	2.771
28	0.256	0.530	0.855	1.313	1.701	2.048	2.467	2.763
29	0.256	0.530	0.854	1.311	1.699	2.045	2.462	2.756
30	0.256	0.530	0.854	1.310	1.697	2.042	2.457	2.750
40	0.255	0.529	0.851	1.303	1.684	2.021	2.423	2.704
60	0.254	0.527	0.848	1.296	1.671	2.000	2.390	2.660
120	0.254	0.526	0.845	1.289	1.658	1.980	2.358	2.617
∞	0.253	0.524	0.842	1.282	1.645	1.960	2.326	2.576

Research 2:

A past study claimed that 3-month-old babies sleep an avarage 20 hours in a 24 hours. A researcher took a random sample of 20 babies and obtained that they slept an avarage 19 hours 15 minutes in a 24 hours with standard deviation of 45 minutes. Assume that the sleeping times of all 3-month-old babies are normally. Using the 5% significance level, test the claim of the earlier study.

X - *the sleeping times of all 3-month-old babies;* $X : N\left(\mu, \sigma^2\right)$

$s = 45\min = 0.75h$ – *sample standard deviation*
σ *unknown – population standard deviation*
$n = 20$ *(sample size)*
$\bar{x} = 19h15\min = 19.25h$ *(sample mean)*
$\alpha = 5\% = 0.05$ *(significance level)*

$$\text{Test-statistics } t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n-1}}} = \frac{19.25 - 20}{\frac{0.75}{\sqrt{20-1}}} \approx -4.36$$

1) $H_0\left(\mu = 20\right)$ *vs* $H_1\left(\mu \neq 20\right)$
two-tailed test;
the rejection region: $C = \left(-\infty, -t_{n-1;\alpha}\right] \cup \left[t_{n-1;\alpha}, \infty\right)$
$t_{n-1;\alpha} = t_{20-1;0.05} = t_{19;0.05} = 2.093 \Rightarrow C = \left(-\infty, -2.093\right] \cup \left[2.093, \infty\right)$
Since $t \approx -4.36 \in C$, *we reject* $H_0\left(\mu = 20\right)$ *- the claim of the earlier study*

2) $H_0\left(\mu = 20\right)$ *vs* $H_1\left(\mu > 20\right)$
right-tailed test;
the rejection region: $C = \left[t_{n-1;2\alpha}, \infty\right)$
$t_{n-1;2\alpha} = t_{20-1;2\cdot 0.05} = t_{19;0.10} = 1.729 \Rightarrow C = \left[1.729, \infty\right)$
Since $t \approx -4.36 \notin C$, *we accept* $H_0\left(\mu = 20\right)$ *- the claim of the earlier study*

3) $H_0\left(\mu = 20\right)$ *vs* $H_1\left(\mu < 20\right)$
left-tailed test;
the rejection region: $C = \left(-\infty, -t_{n-1;2\alpha}\right]$
$t_{n-1;2\alpha} = t_{20-1;2\cdot 0.05} = t_{19;0.10} = 1.729 \Rightarrow C = \left(-\infty, -1.729\right]$
Since $t \approx -4.36 \in C$, *we reject* $H_0\left(\mu = 20\right)$ *- the claim of the earlier study*

Research 3:

A past study claimed that 3-month-old babies sleep an avarage 20 hours in a 24 hours. A researcher took a random sample of 2000 babies and obtained that they slept an avarage 19 hours 15 minutes in a 24 hours with standard deviation of 45 minutes. Using the 5% significance level, test the claim of the earlier study.

Research 4:

A past study claimed that 3-month-old babies sleep an avarage 20 hours in a 24 hours. A researcher took a random sample of 2000 babies and obtained that they slept an avarage 19 hours 15 minutes in a 24 hours. Assume that the population standard deviation is 45 minutes. Using the 5% significance level, test the claim of the earlier study.

Where is the assumption that the sleeping times of all 3-month-old babies are normally?

Central Limit Theorem

If one takes random samples of size **n** *from a population of mean* μ *and standard deviation* σ*, then, as n gets large,* $\overline{X}$ *approaches the normal distribution, that is:* $\overline{X} : N\left(\mu, \frac{\sigma^2}{n}\right)$

X *– random variable – characteristic*
$(X_1, \; X_2, \; X_3, \; \ldots \; X_n)$ *sample*
n – sample size

$$\overline{X} = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

$$E(X_1) = E(X_2) = \ldots = E(X_n) = E(X) = \mu$$

$$\sigma^2(X_1) = \sigma^2(X_2) = \ldots = \sigma^2(X_n) = \sigma^2(X) = \sigma^2.$$

$$E(\overline{X}) = E\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right) = E(X) = \mu,$$

$$\sigma^2(\overline{X}) = \sigma^2\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right) = \frac{\sigma^2(X)}{n} = \frac{\sigma^2}{n}.$$

If $X : N(\mu, \sigma^2)$ *then* $\overline{X} : N\left(\mu, \frac{\sigma^2}{n}\right)$ *, for all n (either small (n<30) or large sample)*

If X has unknown distribution (not normal distribution) and known standard deviation σ *, then* $\overline{X} : N\left(\mu, \frac{\sigma^2}{n}\right)$ *for large sample* $n \geq 30$.

BUT *for CLT, we need the following:*

- ✓ *a large sample size*
- ✓ *known standard deviation* σ

What about the case: a sample is large and σ *is not known?*

Hypothesis tests about the mean μ according to a sample size

1.Small sample

1.1. *σ known*
1.2. *σ not known*

2.Large sample

2.1.*σ known (CLT):* $\overline{X} : N\left(\mu, \frac{\sigma^2}{n}\right)$, *that is* $\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} : N(0,1)$

2.2 .*σ not known: distributon of random variable* $\frac{\overline{X}-\mu}{\frac{\hat{S}}{\sqrt{n}}}$ *can be approximated with normal distribution (as the sample size becomes larger, the t-distribution approaches the standard normal distribution)*

The rejection regions can be obtained in the following way:

- ✓ *two-tailed test*

  $C = (-\infty, -z_\alpha] \cup [z_\alpha, \infty)$ or $C = (-\infty, -t_{n-1;\alpha}] \cup [t_{n-1;\alpha}, \infty)$, *where* $\Phi(z_\alpha) = \frac{1-\alpha}{2}$;

- ✓ *right-tailed test*

  $C = [z_\alpha, \infty)$ or $C = [t_{n-1;2\alpha}, \infty)$, *where* $\Phi(z_\alpha) = \frac{1-2\alpha}{2}$;

- ✓ *left-tailed test*

  $C = (-\infty, -z_\alpha]$ or $C = (-\infty, -t_{n-1;2\alpha}]$, *where* $\Phi(z_\alpha) = \frac{1-2\alpha}{2}$.

Research 3:

A past study claimed that 3-month-old babies sleep an avarage 20 hours in a 24 hours. A researcher took a random sample of 2000 babies and obtained that they slept an avarage 19 hours 15 minutes in a 24 hours with standard deviation of 45 minutes. Using the 5% significance level, test the claim of the earlier study.

X - *the sleeping times of all 3-month-old babies; No assuption about the distribution!!!*

$s = 45\min = 0.75h$ – *sample standard deviation*
σ *unknown – population standard deviation*

(as the sample size becomes larger, the t-distribution approaches the standard normal distribution)

$n = 2000$ *(LARGE sample)*
$\overline{x} = 19h15\min = 19.25h$ *(sample mean)*
$\alpha = 5\% = 0.05$ *(significance level)*

$$\text{Test-statistics } t = \frac{\overline{x} - \mu_0}{\frac{s}{\sqrt{n-1}}} = \frac{19.25 - 20}{\frac{0.75}{\sqrt{2000-1}}} \approx -44.7$$

1) $H_0(\mu = 20)$ *vs* $H_1(\mu \neq 20)$
two-tailed test;
the rejection region: $C = (-\infty, -t_{n-1;\alpha}] \cup [t_{n-1;\alpha}, \infty)$
$t_{n-1;\alpha} = t_{2000-1;0.05} = t_{1999;0.05} = t_{\infty;0.05} = 1.96 \Rightarrow C = (-\infty, -1.96] \cup [1.96, \infty)$
BUT *the rejection region can be also*
$C = (-\infty, -z_\alpha] \cup [z_\alpha, \infty)$ *where* $\Phi(z_\alpha) = \frac{1-\alpha}{2}$
For $\alpha = 0.05$ *we have* $\Phi(z_\alpha) = \frac{1-0.05}{2} = 0.475 \Rightarrow z_\alpha = 1.96 = t_{\infty;0.05}$
Since $t \approx -44.7 \in C$, *we reject* $H_0(\mu = 20)$ *- the claim of the earlier study*

2) $H_0(\mu = 20)$ *vs* $H_1(\mu > 20)$
right-tailed test;
the rejection region: $C = [t_{n-1;2\alpha}, \infty)$
$t_{n-1;2\alpha} = t_{2000-1;2\cdot 0.05} = t_{1999;0.10} = t_{\infty;0.10} = 1.645 \Rightarrow C = [1.645, \infty)$
BUT *the rejection region can be also*
$C = [z_\alpha, \infty)$ *where* $\Phi(z_\alpha) = \frac{1-2\alpha}{2}$
For $\alpha = 0.05$ *we have* $\Phi(z_\alpha) = \frac{1-2\cdot 0.05}{2} = 0.45 \Rightarrow z_\alpha = 1.645 = t_{\infty;0.10}$
Since $t \approx -44.7 \notin C$, *we accept* $H_0(\mu = 20)$ *- the claim of the earlier study*

3) $H_0(\mu = 20)$ *vs* $H_1(\mu < 20)$
left-tailed test;
the rejection region: $C = (-\infty, -t_{n-1;2\alpha}]$
$t_{n-1;2\alpha} = t_{2000-1;2\cdot 0.05} = t_{1999;0.10} = 1.645 \Rightarrow C = (-\infty, -1.645]$
BUT *the rejection region can be also*
$C = (-\infty, -z_\alpha]$ *where* $\Phi(z_\alpha) = \frac{1-2\alpha}{2}$
For $\alpha = 0.05$ *we have* $\Phi(z_\alpha) = \frac{1-2\cdot 0.05}{2} = 0.45 \Rightarrow z_\alpha = 1.645 = t_{\infty;0.10}$
Since $t \approx -44.7 \in C$, *we reject* $H_0(\mu = 20)$ *- the claim of the earlier study*

Research 4:

A past study claimed that 3-month-old babies sleep an avarage 20 hours in a 24 hours. A researcher took a random sample of 2000 babies and obtained that they slept an avarage 19 hours 15 minutes in a 24 hours. Assume that the population standard deviation is 45 minutes. Using the 5% significance level, test the claim of the earlier study.

X - *the sleeping times of all 3-month-old babies;*

No assuption about the distribution!!! CLT!!!

$\sigma = 45\min = 0.75h$ (σ *known*) – *population standard deviation*
$n = 2000$ *(LARGE sample)*
$\bar{x} = 19h15\min = 19.25h$ *(sample mean)*
$\alpha = 5\% = 0.05$ *(significance level)*

$$\text{Test-statistics } t = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{19.25 - 20}{\frac{0.75}{\sqrt{2000}}} \approx -44.72$$

1) $H_0(\mu = 20)$ *vs* $H_1(\mu \neq 20)$
two-tailed test;
the rejection region: $C = (-\infty, -z_\alpha] \cup [z_\alpha, \infty)$

$$\Phi(z_\alpha) = \frac{1 - 0.05}{2} = 0.475 \Rightarrow z_\alpha = 1.96 \Rightarrow C = (-\infty, -1.96] \cup [1.96, \infty)$$

Since $t \approx -44.72 \in C$, *we reject* $H_0(\mu = 20)$ *- the claim of the earlier study*

2) $H_0(\mu = 20)$ *vs* $H_1(\mu > 20)$
right-tailed test;
the rejection region: $C = [z_\alpha, \infty)$

$$\Phi(z_\alpha) = \frac{1 - 2 \cdot 0.05}{2} = 0.45 \Rightarrow z_\alpha = 1.645 \Rightarrow C = [1.645, \infty)$$

Since $t \approx -44.72 \notin C$, *we accept* $H_0(\mu = 20)$ *- the claim of the earlier study*

3) $H_0(\mu = 20)$ *vs* $H_1(\mu < 20)$
left-tailed test;
the rejection region: $C = (-\infty, -z_\alpha]$

$$\Phi(z_\alpha) = \frac{1 - 2 \cdot 0.05}{2} = 0.45 \Rightarrow z_\alpha = 1.645 \Rightarrow C = (-\infty, -1.645]$$

Since $t \approx -44.72 \in C$, *we reject* $H_0(\mu = 20)$ *- the claim of the earlier study*

Example A: *A farmer is supposed to deliver potatoes to a grocery store in packages (bags) that weight 20 kilos (kg) in average. The grocery store claims that the packages are in average under 20 kilos. A random sample of 50 packages of potatoes has an average of 19.4 kilos and standard deviation 1.9 kilos. Test the claim of the store with 1% significance level.*

X - *the weights of farmer's packages;*

σ *unknown – population standard deviation*
$n = 50$ *(large sample)*
$\bar{x} = 19.4kg$ *(sample mean)*
$s = 1.9kg$ *– sample standard deviation*

$\alpha = 1\% = 0.01$ *(significance level)*

$$\text{Test-statistics } t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n-1}}} = \frac{19.4 - 20}{\frac{1.9}{\sqrt{50-1}}} \approx -2.21$$

$$H_0(\mu = 20) \text{ vs } H_1(\mu < 20)$$

left-tailed test;
the rejection region: $C = (-\infty, -t_{n-1;2\alpha}]$
$t_{n-1;2\alpha} = t_{50-1;2\cdot 0.01} = t_{49;0.02} \approx 2.4 \Rightarrow C = (-\infty, -2.4]$
Since $t \approx -2.21 \notin C$, *we accept* $H_0(\mu = 20)$ *- the claim of the farmer*

Notice:
Since the sample is large, the rejection region can be also: $C = (-\infty, -z_\alpha]$ *where* $\Phi(z_\alpha) = \frac{1-2\alpha}{2}$
For $\alpha = 0.01$ *we have* $\Phi(z_\alpha) = \frac{1 - 2\cdot 0.01}{2} = 0.49 \Rightarrow z_\alpha = 2.325 \approx t_{\infty;0.02}$

Example B: *A journalist claims that all adults in her city spend an average of 2 hours or more per week on jogging. A researcher wanted to test this claim. (S)he took a sample of 25 adults from that city and asked them about the time they spend per week on jogging. Their responses are as follows:*

30 min, 1h, 20 min, 0 min, 1h 15min 45 min, 1h, 2h, 2h15mni, 3h,

0 min, 30min, 1h45min, 1h30min, 2h30min, 1h, 2h30min, 3h, 3h30min, 1h,

0min, 15min, 20min, 45min, 1h15min.

Assume that the times spent on jogging per week of all adults from this city are normally distributed. Using the 10% significance level test the claim of the journalist.

X - *the times spent on jogging per week of all adults from the ciy;* $X : N\left(\mu, \sigma^2\right)$

σ *unknown – population standard deviation*

$n = 25$ *(small sample)*

$\bar{x} = ?$

$$\bar{x} = \frac{30+60+20+0+75+45+60+120+135+180+...+20+45+75}{25}\text{min} = 76.6\,\text{min}$$

$\hat{s} = ?\quad s = ?$

$$s^2 = \frac{30^2+60^2+20^2+0^2+75^2+45^2+60^2+120^2+...+20^2+45^2+75^2}{25}\text{min}^2 - 76.6\,\text{min}^2 = 3659.44\,\text{min}^2$$

or

$$s^2 = \frac{(30-76.6)^2+(60-76.6)^2+(20-76.6)^2+...+(45-76.6)^2+(75-76.6)^2}{25}\text{min}^2 = 3659.44\,\text{min}^2$$

$$\hat{s}^2 = \frac{(30-76.6)^2+(60-76.6)^2+(20-76.6)^2+...+(45-76.6)^2+(75-76.6)^2}{24}\text{min}^2 \approx 3811.92\,\text{min}^2$$

$\bar{x} = 76.6\,\text{min}$ *(sample mean)*

$s = \sqrt{3659.44\,\text{min}^2} = 60.49\,\text{min}$ – *sample standard deviation*

$\hat{s} = \sqrt{3811.92\,\text{min}^2} = 61.74\,\text{min}$ - *improved sample standard deviation*

$\alpha = 10\% = 0.10$ *(significance level)*

$$\textit{Test-statistics}\ t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n-1}}} = \frac{76.6 - 120}{\frac{60.49}{\sqrt{25-1}}} \approx -3.515$$

or

$$t = \frac{\bar{x} - \mu_0}{\frac{\hat{s}}{\sqrt{n}}} = \frac{76.6 - 120}{\frac{61.74}{\sqrt{25}}} \approx -3.515$$

$$H_0\left(\mu = 120\right)\ vs\ H_1\left(\mu < 120\right)$$

or

$$H_0\left(\mu \geq 120\right)\ vs\ H_1\left(\mu < 120\right)$$

left-tailed test;

the rejection region: $C = \left(-\infty, -t_{n-1;2\alpha}\right]$

$t_{n-1;2\alpha} = t_{25-1;2\cdot 0.10} = t_{24;0.20} = 1.318 \Rightarrow C = \left(-\infty, -1.318\right]$

Since $t \approx -3.515 \in C$, *we reject* $H_0\left(\mu = 120\right)\left(H_0\left(\mu \geq 120\right)\right)$ - *the claim of the journalist*

Example C: *A recent study claimed that the mean yield per apple plant of sort „G“ is 60 kilos. A researcher has measured the yields of 55 apple plants of certain sort „G“ and obtained the following:*

Yields(kg) per plant	[51, 53)	[53, 55)	[55, 57)	[57, 59)	[59, 61)	[61, 63]
Number of plants	6	9	10	12	10	8

Test the claim of the recent study with 10% significance level

X - *the yields per apple plants of sort „G“*

σ *unknown – population standard deviation*

$n = 55$ *(large sample)*

$\bar{x} = ?$

$$\bar{x} = \frac{6\cdot 52 + 9\cdot 54 + 10\cdot 56 + 12\cdot 58 + 10\cdot 60 + 8\cdot 62}{55} kg \approx 57.27 kg$$

$\hat{s} = ?$ $s = ?$

$$s^2 = \frac{6\cdot 52^2 + 9\cdot 54^2 + 10\cdot 56^2 + 12\cdot 58^2 + 10\cdot 60^2 + 8\cdot 62^2}{55} kg^2 - 57.27^2 kg^2$$

or

$$s^2 = \frac{6\cdot(52-57.27)^2 + 9\cdot(54-57.27)^2 + 10\cdot(56-57.27)^2 + 12\cdot(58-57.27)^2 + 10\cdot(60-57.27)^2 + 8\cdot(62-57.27)^2}{55} kg^2$$

or

$$\hat{s}^2 = \frac{6\cdot(52-57.27)^2 + 9\cdot(54-57.27)^2 + 10\cdot(56-57.27)^2 + 12\cdot(58-57.27)^2 + 10\cdot(60-57.27)^2 + 8\cdot(62-57.27)^2}{54} kg^2$$

$\hat{s}^2 = 9.98 kg^2$

$s^2 \approx 9.80 kg^2$

$\bar{x} = 57.27kg$ *(sample mean)*

$s = 3.13kg$ – *sample standard deviation*

$\hat{s} = 3.16kg$ - *improved sample standard deviation*

$\alpha = 10\% = 0.10$ *(significance level)*

$$\text{Test-statistics } t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n-1}}} = \frac{57.27 - 60}{\frac{3.13}{\sqrt{55-1}}} \approx -6.41$$

or

$$t = \frac{\bar{x} - \mu_0}{\frac{\hat{s}}{\sqrt{n}}} = \frac{57.27 - 60}{\frac{3.16}{\sqrt{55}}} \approx -6.41$$

1) $H_0(\mu = 60)$ *vs* $H_1(\mu \neq 60)$

two-tailed test;

the rejection region: $C = \left(-\infty, -t_{n-1;\alpha}\right] \cup \left[t_{n-1;\alpha}, \infty\right)$

$t_{n-1;\alpha} = t_{55-1;0.10} = t_{54;0.10} \approx 1.671 \Rightarrow C = (-\infty, -1.671] \cup [1.671, \infty)$

Since $t \approx -6.41 \in C$, *we reject* $H_0(\mu = 60)$

$t_{54;0.10} \approx t_{\infty;0.10} = 1.645 \approx z_\alpha;\ \Phi(z_\alpha) = \frac{1-0.10}{2} = 0.45$

2) $H_0(\mu = 60)$ *vs* $H_1(\mu > 60)$

right-tailed test;

the rejection region: $C = \left[t_{n-1;2\alpha}, \infty\right)$

$t_{n-1;2\alpha} = t_{55-1;2\cdot 0.10} = t_{54;0.20} \approx 1.296 \Rightarrow C = [1.296, \infty)$

Since $t \approx -6.41 \notin C$, *we accept* $H_0(\mu = 60)$

$t_{54;0.20} \approx t_{\infty;0.20} = 1.282 \approx z_\alpha;\ \Phi(z_\alpha) = \frac{1-0.20}{2} = 0.40$

3) $H_0(\mu = 60)$ *vs* $H_1(\mu < 60)$

left-tailed test;

the rejection region: $C = \left(-\infty, -t_{n-1;2\alpha}\right]$

$t_{n-1;2\alpha} = t_{55-1;2\cdot 0.10} = t_{54;0.20} = 1.296 \Rightarrow C = (-\infty, -1.296]$

Since $t \approx -6.41 \in C$, *we reject* $H_0(\mu = 60)$

Example D: *The visitors at the museum complained about long lines and the time they had to spend waiting to buy entrance tickets. A random sample of 10 visitors of that museum has been chosen and they have been asked about the time spent in line waiting to buy entrance tickets. Their responses in minutes are as follows:*

5min, 10min, 15min, 20min, 5min, 10min, 30min, 15min, 20min, 10min

Assume that the times spent in line waiting to buy entrance tickets for this museuem are normally distributed and population standard deviation is 5 minutes. Test the claim that the entrance ticket for this museum can be bought in 10 minutes on average with 5% significance level.

X - *the times spent in the line waiting to buy entrance ticket for the museum;* $X : N\left(\mu,\sigma^2\right)$

$\sigma = 5\,\text{min}$ (σ *known*) – *population standard deviation*
$n = 10$ *(small sample)*
$\alpha = 5\% = 0.05$ *(significance level)*

$\bar{x} = ?$ *(sample mean)*

$$\bar{x} = \frac{5+10+15+20+5+10+30+15+20+10}{10}\,\text{min} = 14\,\text{min}$$

$$\text{Test-statistics } t = \frac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{14-10}{\frac{5}{\sqrt{10}}} \approx 2.53$$

$$H_0\left(\mu = 10\right) \text{ vs } H_1\left(\mu > 10\right)$$

right-tailed test;
the rejection region: $C = \left[z_\alpha, \infty\right)$

$\Phi\left(z_\alpha\right) = \frac{1-2\cdot 0.05}{2} = 0.45 \Rightarrow z_\alpha = 1.645 \Rightarrow C = \left[1.645, \infty\right)$

Since $t \approx 2.53 \in C$, *we reject* $H_0\left(\mu = 10\right)$